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Two equal symmetric circular arc cracks in an elastic medium – the Dugdale approach

R. R. BHARGAVA and RAJESH KUMAR
Department of Mathematics, University of Roorkee, Roorkee-247 667, India

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Abstract. The Dugdale model for two equal, symmetrically situated coplanar circular arc cracks contained in an infinite elastic perfectly-plastic plate is proposed. Biaxial loads are applied at the infinite boundary of the plate. Consequently, the rims of the cracks open in Mode I and develop a plastic zone ahead of each of the cracks. These plastic zones are then closed by the distribution of uniform normal closing stresses over the rims of the plastic zones. Based on the complex-variable technique and the superposition principle, the solution for the above problem is obtained. Closed-form analytic expressions are obtained for the determination of the sizes of the plastic zones and the crack-opening displacement (COD) at the tip of the crack. Numerical studies are carried out to calculate the load ratio (load applied at infinity/yield point stress applied at the rims of the plastic zones) required for the closure of the plastic zones, for various radii of arc cracks and for various angles subtended by them at the centre. The crack-opening displacement is also investigated with respect to these parameters.

Key words: cracks, plate, plasticity, elasticity, Dugdale.

1. Introduction

The problem of stress distribution in the neighbourhood of two equal collinear Griffith cracks subjected to a uniform normal pressure was first considered by Wilmore [1]. Tranter [2] extended this problem by varying pressure along the crack length. Panayasuk and Lozovoi [3] determined the limiting stresses in an elastic plane with two unequal straight cracks under tension perpendicular to the length of the cracks. The solution for the problem of two cracks contained in an infinite plate under various loads was given by England and Green [4], Sneddon and Srivastava [5], Lowengrub and Srivastava [6] and others.

A powerful complex-variable approach for arc cracks was developed by Muskhelishvili [7] for the two-dimensional theory of elasticity. The stress-intensity factor at the tip of a circular arc crack has been investigated by Panayasuk [8] *et al.* The problem of two coplanar arc cracks under shear load has been considered by Fu and Keer [9] using integral equations. Thermal stresses were calculated at the tips of the parallel circular arc cracks by Kassir and Bregman [10]. The behaviour or two concentric circular arc cracks contained in an infinite plane under anti-plane strain conditions has been analyzed by Jagannadham [11]. More recently, the elastic problem for two circular arc cracks contained in an infinite elastic plate has been studied by Piva and Viola [12] and Gdoutos [13] *et al.*

It is observed that, when a cracked sheet is subjected to tensile loads normal to the rims of a straight crack, then it opens, developing plastic zones ahead of the tips of the crack. Dugdale [14] suggested a model in which these plastic zones were closed by distributing cohesive yield point stress over them. Theocaris [15] extended the Dugdale model to determine the size of the plastic zones which develop ahead of the tips of two collinear straight cracks contained in an infinite plate under conditions of plane stress.

In the present paper the following problem is investigated: Consider an infinite elastic perfectly-plastic plate containing two coplanar, equal and symmetrically situated circular arc cracks lying on the same circle. The infinite plate is subjected to a biaxial load applied at infinity. Consequently, the rims of cracks open in Mode I developing plastic zones ahead of the tips of the cracks. The plastic zones are, in turn, subjected to a cohesive yield point stress which closes the plastic zones, thus arresting the cracks. Closed-form expressions are obtained through the application of a complex variable technique. The size of the plastic zone and the crack-face opening displacement are obtained.

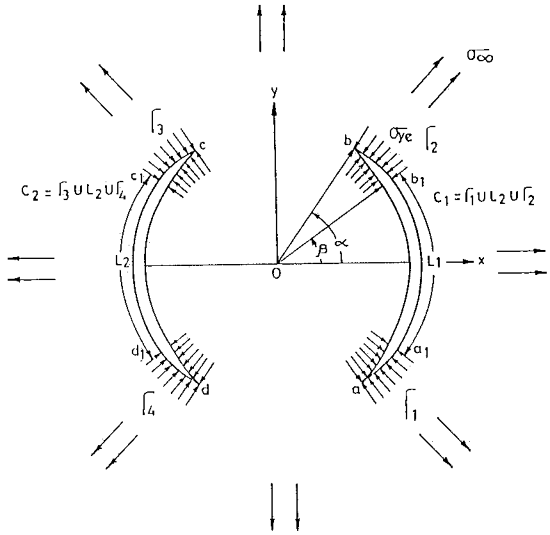

Figure 1. Configuration.

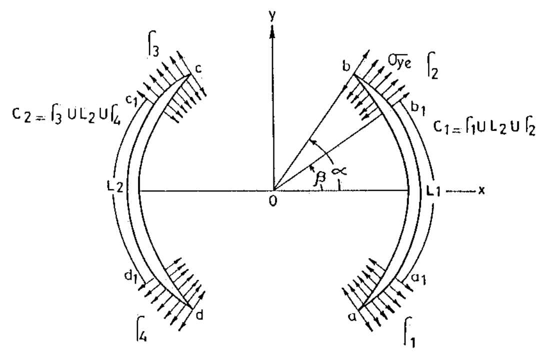

Figure 2. Configuration of Problem II.

2. Formulation and solution of the problem

A homogeneous, isotropic, elastic perfectly-plastic infinite plate, which occupies the xy-plane (shown in Figure 1), contains two coplanar, equal and symmetrically situated circular arc cracks L_1 and L_2. These cracks lie on a circle of radius R with centre (0,0). The crack L_1 lies from $a_1(= R\,\mathrm{e}^{-i\beta})$ to $b_1(= R\,\mathrm{e}^{i\beta})$ and the crack L_2 from $c_1(= -R\,\mathrm{e}^{-i\beta})$ to $d_1(= -R\,\mathrm{e}^{i\beta})$ as shown in Figure 1. The configuration so obtained is subjected to tensile biaxial stresses at infinity. On account of these loads the plastic zones $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 develop ahead of the tips a_1, b_1, c_1 and d_1 of the cracks L_1 and L_2, respectively. These plastic zones $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 lie in the intervals $[a, a_1]\{= [R\,\mathrm{e}^{-i\alpha}, R\,\mathrm{e}^{-i\beta}]\}, (b_1, b)\{= [R\,\mathrm{e}^{i\beta}, R\,\mathrm{e}^{i\alpha}]\}, [c, c_1]\{= [-R\,\mathrm{e}^{i\alpha}, -R\,\mathrm{e}^{i\beta}]\}$, and $[d_1, d]\{= [-R\,\mathrm{e}^{i\beta}, -R\,\mathrm{e}^{-i\alpha}]\}$, respectively. Each of the rims of the plastic zones $\Gamma_i (i = 1, 2, 3, 4)$ is subjected to a uniform cohesive yield point stress, σ_{ye}, causing their closure. The remaining part of the crack rims are stress free. The entire configuration of the problem is depicted in Figure 2. The boundary conditions of the problem may be stated as follows:

(a) At infinity a uniform tensile biaxial stress, σ_∞, is prescribed everywhere. Thus, a stress $\sigma_\infty = P_{rr}$ may be imagined to act on the rims of the circular boundary.

(b) The circular boundary may be considered stress free and $P_{rr}(= -\sigma_\infty)$ is applied at the rims of the cracks.

(c) The above stresses in (b) give rise to plastic zones $\Gamma_i (i = 1, 2, 3, 4)$ at the tips of the cracks.

The solution of the above problem is obtained by superposition of the solutions of two component problems contributing to the stress singularity at the tips of the cracks. These problems are designated as problem I and problem II. These are formulated and solved by a complex-variable approach in the next two sections.

3. Problem I

This problem may be stated as follows. An infinite, homogeneous, isotropic, elastic perfectly-plastic plate lying in the xy plane (as shown in Figure 2), contains two equal, coplanar and symmetrically situated circular arc cracks $C_1\{= \Gamma_1 \cup L_1 \cup \Gamma_2\}$ and $C_2\{= \Gamma_3 \cup L_2 \cup \Gamma_4\}$. The crack C_1 lies from $R\,\mathrm{e}^{-i\alpha}$ to $R\,\mathrm{e}^{i\alpha}$ and the crack C_2 from $R\,\mathrm{e}^{i\alpha}$ to $-R\,\mathrm{e}^{-i\alpha}$. Boundary conditions of the problem are:

(i) No stresses are acting at infinity.
(ii) The rims of the cracks C_1 and C_2 are opened by the application of a uniform tensile stress σ_∞.

The solution of this classical problem may be written down diirectly, when we use the equations from (A6) to (A13) of Appendix A, with $p(t) = \sigma_\infty$ and $q(t) = 0$. The complex potential $\phi_1(z)$, (the subscript 1 denotes the potential referring to problem I) may be written as (with Ci and Di such that $\phi_1(z) = o(z^{-2})$, for $z \to \infty$)

$$\phi_1(z) = \frac{\sigma_\infty}{\varepsilon - 2}\left[1 - \frac{1}{X(z)}\{z^2 + R^2(1 - 2\varepsilon)\}\right], \tag{1}$$

where

$$X(z) = \{(z^2 - R^2\,\mathrm{e}^{-2i\alpha})(z^2 - R^2\,\mathrm{e}^{2i\alpha})\}^{1/2}, \tag{2}$$

$$\varepsilon = E/F, \quad E = E(\pi/2, \sin\alpha), \quad F = F(\pi/2, \sin\alpha) \tag{2a}$$

are complete elliptic integrals of first and second kinds, respectively.

4. Problem II

A stress-free infinite, homogeneous, isotropic, elastic-perfectly plastic plate, situated in the xy plane, contains two coplanar, equal and symmetrically situated circular arc cracks $C_1\{= \Gamma_1 \cup L_1 \cup \Gamma_2\}$ and $C_2\{= \Gamma_3 \cup L_2 \cup \Gamma_4\}$ as shown in Figure 2. The cracks $C_i (i = 1, 2)$ are formed with the union of the actual crack $L_i (i = 1, 2)$ and relevant plastic zones $\Gamma_i (i = 1, 2, 3, 4)$ existing ahead of crack tips. The boundary conditions of the problem are

(1) No stresses are acting at infinity (so, $\phi_2(z) = o(z^{-2}), z \to \infty)$.
(2) Each of the plastic zones $\Gamma_i (i = 1, 2, 3, 4)$ is subjected to a uniform normal yield point stress, σ_{ye}, as shown in Figure 2, and the remaining part of the rims of the cracks C_1 and C_2 are stress free (yielding $p(t) = \sigma_{ye}, q(t) = 0$, on Γ_i).

Using boundary condition (2) above and equation (A6) of the Appendix A, we obtain the following Hilbert problems

$$\phi^{+} +_2 (t) + \Omega_2^{-}(t) = \sigma_{ye}, \qquad \phi_2^{-}(t) + \Omega_2^{+}(t) = \sigma_{ye}, \quad \text{on } L = \overset{4}{\underset{1}{\varphi}} \Gamma_i, \tag{3}$$

where any point on L is denoted by $t = R\,\mathrm{e}^{i\theta}$. The superscripts $+$ and $-$ refer to the values of the function at t as it is approached from the 'inner' $(r < R)$ and from the 'outer' $(r > R)$ region of the crack. Subscript 2 indicates that the potential refers to problem II.

The solution of Equations (3) for potential $\phi_2(z)$ may be obtained from the Equations (A8) to (A13) of Appendix A.

$$\begin{aligned} \phi_2(z) &= \frac{\sigma_{ye}}{\pi i X(z)}[R^2 S_1 + (z^2 - R^2 \cos 2\alpha) S_2 + 2iX(z) S_3] \\ &\quad + \frac{1}{2X(z)}(C_0 z^2 + C_2) + \tfrac{1}{2} D_0, \end{aligned} \tag{4}$$

where $X(z)$ is given by Equation (2) and

$$S_1 = \sqrt{(\mathrm{e}^{-2i\beta} - \mathrm{e}^{-2i\alpha})(\mathrm{e}^{-2i\beta} - \mathrm{e}^{2i\alpha})} - \sqrt{(\mathrm{e}^{2i\beta} - \mathrm{e}^{-2i\alpha})(\mathrm{e}^{2i\beta} - \mathrm{e}^{2i\alpha})}, \tag{5}$$

$$\begin{aligned} S_2 &= \log \left[\frac{\sqrt{\{(\mathrm{e}^{-2i\beta} - \mathrm{e}^{-2i\alpha})(\mathrm{e}^{-2i\beta} - \mathrm{e}^{2i\alpha})\}} + \mathrm{e}^{2i\beta} - \cos 2\alpha}{\sqrt{\{(\mathrm{e}^{2i\beta} - \mathrm{e}^{2i\alpha})(\mathrm{e}^{2i\beta} - \mathrm{e}^{2i\alpha})\}} + \mathrm{e}^{2i\beta} - \cos 2\alpha} \right] \\ &\quad + \ln \left[\frac{\mathrm{e}^{2i\alpha} - \cos 2\alpha}{\mathrm{e}^{-2i\alpha} - \cos 2\alpha} \right], \end{aligned} \tag{6}$$

$$\begin{aligned} S_3 &= \tan^{-1} \sqrt{\frac{(R^2\,\mathrm{e}^{-2i\alpha} - z^2)(\mathrm{e}^{-2i\beta} - \mathrm{e}^{2i\alpha})}{(z^2 - R^2\,\mathrm{e}^{2i\alpha})(\mathrm{e}^{-2i\beta} - \mathrm{e}^{-2i\alpha})}} \\ &\quad - \tan^{-1} \sqrt{\frac{(R^2\,\mathrm{e}^{-2i\alpha} - z^2)(\mathrm{e}^{2i\beta} - \mathrm{e}^{2i\alpha})}{(z^2 - R^2\,\mathrm{e}^{2i\alpha})(\mathrm{e}^{2i\beta} - \mathrm{e}^{-2i\alpha})}} - \tfrac{1}{2}\pi \end{aligned} \tag{7}$$

$$C_0 = 2\sigma_{ye} \frac{(E - E_1 + F_1 - F)(F - F_1)}{(2F - 2F_1 - E + E_1)(E - E_1)} \left\{ \frac{1}{\pi}(S_5 - S_4) + \frac{S_2}{\pi i} \left(\frac{E - E_1}{F - F_1} \right) \right\}, \tag{8}$$

where $E = E(\pi/2, \sin\alpha)$ and $F = F(\pi/2, \sin\alpha)$ are complete elliptic integrals as stated in Equation (2a) and $F_1 = F(\kappa, k)$ and $E_1 = E(\kappa, k)$ are incomplete elliptic integrals of the first and second kinds, respectively, and

$$\kappa = \sin^{-1}\left(\frac{\sin\beta}{\sin\alpha}\right), \qquad k = \sin\alpha, \tag{9}$$

$$C_2 = -\frac{2\sigma_{ye}R^2}{\pi i}\left[(S_1 - \cos 2\alpha \cdot S_2) + \frac{(2E - 2E_1 + F_1 - F)}{(F - F_1)} \cdot S_2 + \frac{(2E - 2E_1 + F_1 - F)(E - E_1 + F_1 - F)}{(2F - 2F_1 - E + E_1)(E - E_1)} \times \left\{(S_3 - S_4)i + \frac{(E - E_1)}{(F - F_1)}S_2\right\}\right]. \tag{10}$$

We also have

$$A^2 = \frac{(\mathrm{e}^{2i\alpha} - \mathrm{e}^{-2i\beta})}{(\mathrm{e}^{-2i\beta} - \mathrm{e}^{-2i\alpha})} \quad \text{and} \quad B^2 = \frac{(\mathrm{e}^{2i\alpha} - \mathrm{e}^{2i\beta})}{(\mathrm{e}^{2i\beta} - \mathrm{e}^{-2i\alpha})}, \tag{11}$$

$$S_4 = \tan^{-1} A - \tan^{-1} B - \tfrac{1}{2}\pi, \tag{12}$$

$$S_5 = \tan^{-1}(\mathrm{e}^{-2i\alpha} A) - \tan^{-1}(\mathrm{e}^{-2i\alpha} B) - \tfrac{1}{2}\pi, \tag{13}$$

$$D_0 = -C_0 - \frac{2\sigma_{ye}}{\pi i}(S_2 + i2S_4). \tag{14}$$

We obtain the potential $\Omega_2(z)$ by substituting for $\phi_2(z)$ and D_0 from Equations (4) and (14) in Equation (A8). Note that $q(t)$ in Equation (A11) is identically zero.

5. Plastic-zone size and crack-opening displacement

According to Dugdale's strip yield model the stresses should remain finite at every point of the body. Consequently, the stress-intensity factors at the tips of the cracks for problem I (Section 3) and problem II (Section 4) must cancel each other. Because of the symmetry of the problem it suffices to calculate the stress-intensity factors at only one tip of the cracks (say) $z = b = R\,\mathrm{e}^{i\alpha}$.

For the problem I, the normal stress-intensity factor $K_{11} = R\,\mathrm{e}[K]$ at the crack tip $z = b = R\,\mathrm{e}^{i\alpha}$ is obtained by substituting $\phi_1(z)$ from Equation (1) in Equation (A14) of Appendix A

$$K_{11} = \mathrm{Re}\left[\frac{\sigma_\infty}{2 - \varepsilon}\sqrt{\frac{2\pi R}{i\,\mathrm{e}^{i\alpha}\,\sin 2\alpha}}\left\{\mathrm{e}^{2i\alpha} + (1 - 2\varepsilon)\right\}\right], \tag{15}$$

The normal stress-intensity factor, $K_{21} = \mathrm{Re}[K]$ at the tip $z = b = R\,\mathrm{e}^{i\alpha}$ is calculated substituting $\phi_2(z)$ for $\phi(z)$ from Equation (4) in Equation (A14), for the problem II. It is given by the expression

$$K_{21} = \mathrm{Re}\left[\sqrt{\frac{2\pi R}{i\,\mathrm{e}^{i\alpha}\,\sin 2\alpha}}\left\{\frac{\sigma_{ye}}{\pi}(\sin 2\alpha S_2 + iS_1) + \tfrac{1}{2}(C_0\,\mathrm{e}^{2i\alpha} + C_2/R^2)\right\}\right]. \tag{16}$$

At the tip $z = b = R\,\mathrm{e}^{i\alpha}$, these stress intensity-factors balance each other *i.e.*

$$K_{11} = K_{21} \tag{17}$$

giving a nonlinear equation in terms of parameters $\sigma_\infty/\sigma_{ye}, \alpha, \beta$ and R. For prescribed $\sigma_\infty/\sigma_{ye}, \beta$ and R the unknown α may be calculated from this equation. The plastic zone length is then calculated as $R(\alpha-\beta)$

The crack-opening displacement, u_r, along the radius of the crack, at the actual crack tip $z=b_1=R\,\mathrm{e}^{i\beta}$ is obtained by substitution of $\phi_1(t)$, for problem 1, for $\phi(t)$ in Equation (A17) and then by integration.

The expression for COD $(u_r)_1$ at $z=b_1=R\,\mathrm{e}^{i\beta}$ may finally be written as

$$(u_r)_1 = \mathrm{Re}\left[\frac{-\sigma_\infty\,\mathrm{e}^{-i\beta}(\kappa+1)}{i\mu(2-\varepsilon)}(E_1-\varepsilon F_1 - i\,\sin\,\alpha\,\cos\,\phi)\right]. \tag{18}$$

The subscript 1 after (u_r) indicates the COD corresponds to the problem I. For problem II, we obtain the crack-opening displacement, $(u_r)_2$, at the actual crack tip $z=b_1=R\,\mathrm{e}^{i\beta}$ of the crack L_1 similarly by substituting the value of $\phi_2(t)$ from Equation (4) in (A17) and integrating. We get

$$\begin{aligned}(u_r)_2 \;=\; \mathrm{Re}\Bigg[&\frac{-R\,\mathrm{e}^{-i\beta}(\kappa+1)}{4i\mu}\Bigg[\frac{2}{\pi i}\sigma_{ye}\{S_1F_1+S_2(2E_1-F_1-2i\,\sin\,\alpha\,\cos\,\phi)\\ &-\cos\,2\alpha\cdot S_2\cdot F_1\}+C_0(2E_1-F_1-2i\,\sin\,\alpha\,\cos\,\phi)+F_1C_2/R^2\Bigg]\Bigg]. \end{aligned}\tag{19}$$

The subscript 2 after (u_r) denotes that the COD corresponds to problem II.

The Dugdale model crack-opening displacement for the original formulated problem in Section 2 is then calculated as in [17] by

$$u_r = (u_r)_1\frac{\sigma_\infty}{\sigma_{ye}} - (u_r)_2, \tag{20}$$

substitution of $(u_r)_1$ from Equation (18) and $(u_r)_2$ from Equation (19), $\sigma_\infty/\sigma_{ye}$ from Equation (17). Note that for a plastic zone size, crack length and crack radius are taken as calculated from the above equations.

6. Numerical calculation and results

We study the qualitative behaviour of load ratio (load applied at infinity/yield stress applied at the plastic zones) and corresponding crack-opening displacement, using the above analysis. Other parameters are crack radius R, crack length, inter-crack distance (the arcual distance between two neighbouring tips of the cracks L_1 and L_2) and plastic-zone size. Some illustrative numerical examples are considered. The loaded boundary was taken at sufficiently large distances. The results obtained are reported graphically.

Figure 3 shows the variation of the load ratio, $\sigma_\infty/\sigma_{ye}$, as the length of the plastic zone is increased. The calculations are carried out for a fixed half crack angle, $\beta=45°$. As expected, the required load ratio increases with the increase in plastic-zone size. The four curves show that, as the inter-crack distance is increased, the load required for closure reduces, as expected, for a fixed plastic-zone size and crack length. This result matches with a similar type of study carried out by Theocaris [16].

Variation of the load ratio versus β (half-crack angle), for a fixed crack radius, $R = 3$, is drawn in Figure 4. It may be noted that for a fixed plastic zone, as the crack length is increased, the required load ratio for closure also reduces, as expected. It is also observed that increasing the length of the crack, the load required for closure reduces for a fixed plastic zone, as expected. If the size of plastic zone is increased, the load required for arresting the cracks is also increased.

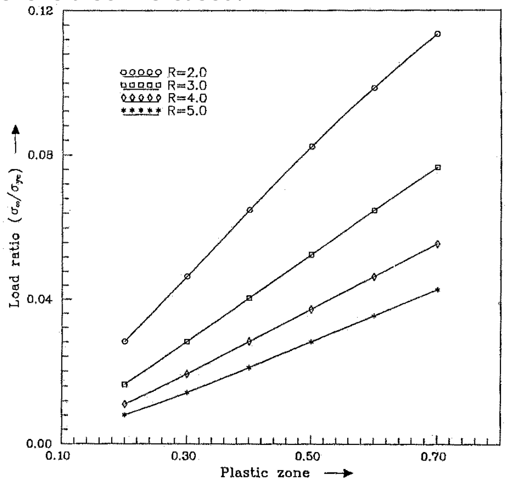

Figure 3. Variation of required load ratio versus plastic zone as the inter crack distance increases, for half crack angle $\beta = 45°$.

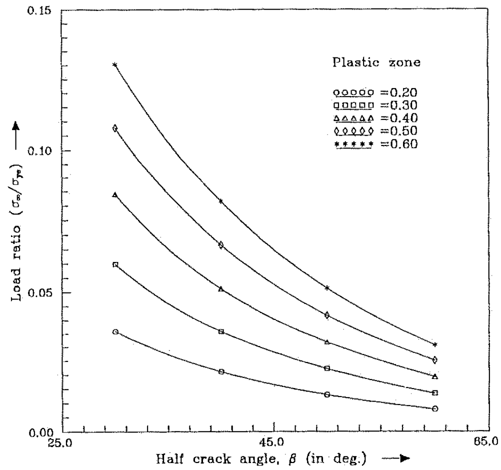

Figure 4. Variation of load ratio versus half crack angle for radius $R = 3$.

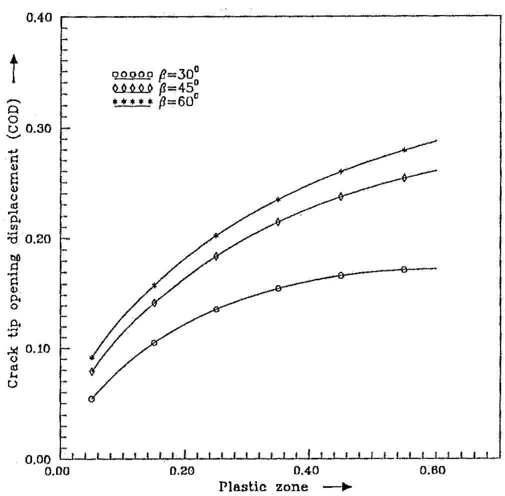

Figure 5. Variation of crack tip opening displacement versus plastic zone for $R = 3$.

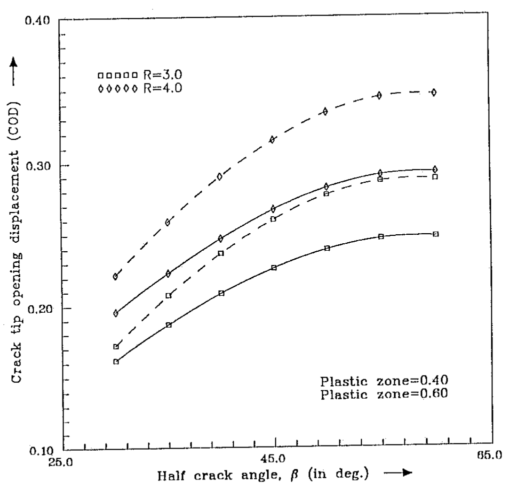

Figure 6. Variation of crack tip opening displacement versus inter crack distance for fixed crack length $\beta = 45°$.

Figure 5 depicts that the crack will open more if the size of the plastic zone is increased. The normalized crack opening displacement at the tip $z = b$ of the crack has been plotted in this figure for $R = 3$ and for three different values of $\beta = 30°$, $45°$ and $60°$. We observe that, increasing the half crack angle, the crack opens more for a prescribed load ratio.

For a fixed half crack angle $\beta = 45°$, the variation of crack opening displacement (at the tip $z = b_1 = R\,\mathrm{e}^{i\beta}$ of the crack L_1) against inter-crack distance is plotted in Figure 6. It is observed that, as the crack size is increased, it opens more for a fixed plastic-zone size. Also, it may be noted that as plastic zone size is increased, the cracks open still more.

Appendix A

Complex variable formulation and solution before According to the complex-variable technique developed by Muskhelishvili [7], the stress components $P_{ij}(i, j = r, \theta)$ may be expressed in terms of two complex potentials $\phi(z)$ and $\psi(z)$ as

$$P_{rr} + iP_{r\theta} = \phi(z) + \overline{\phi(z)} - \overline{z\phi'(z)} - (\bar{z}/z)\overline{\psi(z)}, \tag{A1}$$

$$\mu\frac{\partial}{\partial\theta}\{\mathrm{e}^{i\theta}(u_r + iu_\theta)\} = \tfrac{1}{2}iz\{\kappa\phi(z) - \overline{\phi(z)} + \overline{z\phi'(z)} + (\bar{z}/z)\overline{\psi(z)}\}, \tag{A2}$$

where $z = r\,\mathrm{e}^{i\theta}$, μ is the shear modulus, $\kappa = 3 - 4\nu$ for plane-strain case; $\kappa = (3-\nu)/(1+\nu)$ for the generalized plane-stress case and ν is Poisson's ratio. The bar over the function denotes its complex conjugate.

For the problem of an infinite plate, cut along circular arcs of one and the same circle, instead of $\psi(z)$, a new potential $\Omega(z)$, related to $\phi(z)$ and $\psi(z)$, is found to be more convenient. The relation between $\Omega(z)$, $\phi(z)$ and $\psi(z)$ is given by

$$\Omega(z) = \bar{\phi}\left(\frac{R^2}{z}\right) - \frac{R^2}{z}\bar{\phi}'\left(\frac{R^2}{z}\right) - \frac{R^2}{z^2}\bar{\psi}\left(\frac{R^2}{z}\right), \tag{A3}$$

where R is the radius of the circle on which the cracks lie.

Substituting value of $\psi(z)$ from Equation (A3) in Equation (A1) and (A2), we may now express the stress and displacement components as

$$P_{rr} + iP_{r\theta} = \phi(z) + \Omega\left(\frac{R^2}{\bar{z}}\right) + \bar{z}\left(\frac{\bar{z}}{R^2} - \frac{1}{z}\right)\overline{\psi(z)}, \tag{A4}$$

and

$$2\mu\frac{\partial}{\partial\theta}\{\mathrm{e}^{i\theta}(u_r + iu_\theta)\} = iz\left\{\kappa\phi(z) - \Omega\left(\frac{R^2}{\bar{z}}\right)\right\} - \bar{z}\left(\frac{\bar{z}}{R^2} - \frac{1}{z}\right)\overline{\psi(z)}. \tag{A5}$$

Consider an infinite elastic plate cut along the arc L_k (with end points a_k, b_k), $(k = 1, 2, \ldots, n)$ of one and the same circle of radius R and centre (0,0). The rims of the crack $L\{= \cup L_k, k = 1, 2, \ldots, n\}$ are subjected to prescribed stress $P_{rr}^{\pm} + iP_{r\theta}^{\pm}$. The superscript $+$ and $-$ refer to the values of the function at t as it is approached from the 'inner' $(r < R)$ and from the 'outer' $(r > R)$ region of the crack, respectively. Using Equation (A4), we obtain the following Hilbert problems

$$\begin{aligned} &\phi^+(t) + \Omega^-(t) = P_{rr}^+ + ip_{r\theta}^+, \\ &\phi^-(t) + \Omega^+(t) = P_{rr}^- + ip_{r\theta}^-, \end{aligned} \tag{A6}$$

under the condition

$$\lim_{r->R} \left\{ \mathrm{e}^{-i\theta} \left(\frac{r}{R^2} - \frac{1}{r} \right) \psi(z) \right\} = 0 \tag{A7}$$

and any point on L is denoted by $t = R\,\mathrm{e}^{i\theta}$.

The solution of (A6) in the absence of body forces and zero stresses at infinity may be written as

$$\phi(z) - \Omega(z) = \frac{1}{\pi i} \int_L \frac{q(t)}{t - z}\,\mathrm{d}t + D_0 \tag{A8}$$

and

$$\phi(z) + \Omega(z) = \frac{1}{\pi i} \int_L \frac{X(t)p(t)}{t - z}\,\mathrm{d}t + \frac{1}{X(z)} \left\{ P_n(z) + \frac{D_1}{z} + \frac{D_2}{z^2} \right\}, \tag{A9}$$

where

$$X(z) = \prod_{k=1}^{n} (z - a_k)^{1/2} (z - b_k)^{1/2}, \tag{A10}$$

$$q(t) = \tfrac{1}{2}(P_{rr}^{+} - P_{rr}^{-}) + \tfrac{1}{2} i (P_{r\theta}^{+} - P_{r\theta}^{-}), \tag{A11}$$

$$p(t) = \tfrac{1}{2}(P_{rr}^{+} + P_{rr}^{-}) + \tfrac{1}{2} i (P_{r\theta}^{+} + P_{r\theta}^{-}). \tag{A12}$$

$$P_n(z) = c_0 z^n + c_1 z^{n-1} + \cdots + c_n. \tag{A13}$$

The constants D_0, D_1, D_2 and $C_i (i = 1, 2, \ldots, n)$ are determined from the boundary conditions of the problem under consideration and single valuedness of the displacements at the crack rims.

The stress intensity factor at the crack tip $z = z_1$ is determined from the relation [18, pp. 97]

$$K_1 - iK_2 = 2\sqrt{2\pi} \lim_{z \to \mathrm{Z}_1} \{(z - z_1)^{1/2} \phi(z)\}. \tag{A14}$$

The relative crack-face opening displacement of crack face, $V(t)$, may be defined as

$$V(t) = v^{-}(t) - v^{+}(t), \tag{A15}$$

where $v^{-}(t) = \lim_{z \uparrow t} v(z); v^{+}(t) = \lim_{z \downarrow t} v(z)$ and $t = R\,\mathrm{e}^{i\theta}$. Introducing

$$v(z) = \mathrm{e}^{i\theta} \{(u_r(z) + i u_\theta(z)\} \tag{A16}$$

and substituting values of $\phi(z)$ and $\Omega(z) = \phi(z) - D_0$ (since $q(t) = 0$) in (A5), we obtain

$$V'(t) = \frac{\kappa + 1}{2\mu} \{\phi^{-}(t) - \phi^{+}(t)\}. \tag{A17}$$

$V(t)$ is then obtained by integration of Equation (A17), and hence u_r, the crack-face opening displacement, is obtained.

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